# Bifurcations and sudden current change in ensembles of classically chaotic ratchets

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(Received 10 August 2006; published 30 May 2007)

Mateos [Phys. Rev. Lett. **84**, 258 (2000)] conjectured that current reversal in a classical deterministic ratchet is associated with bifurcations from chaotic to periodic regimes. This is based on the comparison of the current and the bifurcation diagram as a function of a given parameter for a periodic asymmetric potential. Barbi and Salerno [Phys. Rev. E **62**, 1988 (2000)] have further investigated this claim and argue that, contrary to Mateos' claim, current reversals can occur also in the absence of bifurcations. Barbi and Salerno's studies are based on the dynamics of one particle rather than the statistical mechanics of an ensemble of particles moving in the chaotic system. The behavior of ensembles can be quite different, depending upon their characteristics, which leaves their results open to question. In this paper we present results from studies showing how the current depends on the details of the ensemble used to generate it, as well as conditions for convergent behavior (that is, independent of the details of the ensemble). We are then able to present the converged current as a function of parameters, in the same system as Mateos as well as Barbi and Salerno. We show evidence for current reversal without bifurcation, as well as bifurcation without current reversal. We conjecture that it is appropriate to correlate abrupt changes in the current with bifurcation, rather than current reversals, and show numerical evidence for our claims.

DOI: 10.1103/PhysRevE.75.056215 PACS number(s): 05.45.−a

## I. INTRODUCTION

The transport properties of nonlinear nonequilibrium dynamical systems are far from well-understood [1]. Consider in particular so-called ratchet systems which are asymmetric periodic potentials where an ensemble of particles experience directed transport [2,3]. The origins of the interest in this lie in considerations about extracting useful work from unbiased noisy fluctuations as seems to happen in biological systems [4–6]. Based on the pioneering analysis of Ref. [7], recent attention has been focused on the behavior of deterministic chaotic ratchets [8–12] as well as Hamiltonian ratchets [13,14].

Chaotic systems are defined as those which are sensitively dependent on initial conditions. Whether chaotic or not, the behavior of nonlinear systems—including the transition from regular to chaotic behavior—is in general sensitively dependent on the parameters of the system. That is, the phase-space structure is usually relatively complicated, consisting of stability islands embedded in chaotic seas, for example, or of simultaneously coexisting attractors. This can change significantly as parameters change. For example, stability islands can merge into each other, or break apart, and the chaotic sea itself may get pinched off or otherwise changed, or attractors can change symmetry or bifurcate. This means that the transport properties can change dramatically as well. A few years ago, Mateos [8] considered a periodically forced underdamped ratchet model, based on work in Ref. [7]. He looked at the velocity for an ensemble of particles averaged over time and the entire ensemble. He showed that this quantity, an intuitively reasonable definition of "the current," could be either positive or negative depending on the amplitude $a$ of the periodic forcing for the system. At the same time, there exist ranges in $a$ where the trajectory of an individual particle displays chaotic dynamics. Mateos conjectured that the reversal of current direction was correlated with a bifurcation from chaotic to periodic behavior in the trajectory dynamics. Even though it is unlikely that such a result would be universally valid across all chaotic deterministic ratchets, it would still be extremely useful to have general heuristic rules such as this, allowing some characterization of the many different kinds of behavior that are possible.

A later investigation [9] of the Mateos conjecture by Barbi and Salerno, however, argued that it was not a valid rule even for the specific system considered by Mateos. They presented results showing that it was possible to have current reversals in the absence of bifurcations from periodic to chaotic behavior. They proposed an alternative origin for the current reversal, suggesting it was related to the different stability properties of the rotating periodic orbits of the system. These latter results seem fundamentally sensible. However, their paper based its arguments about currents on the behavior of a *single* particle as opposed to an ensemble. This implicitly assumes that the dynamics of the system are ergodic; there was a further invalid assumption about phase-locking of the dynamics in computing the current. These assumptions do not hold in general for chaotic systems of the type being considered. In particular, there can be extreme dependence of the result on the statistics of the ensemble being considered. This has been pointed out in earlier studies [7,8] which laid out a detailed methodology for understanding transport properties in such a mixed regular and chaotic system. Depending on specific parameter value, the particular system under consideration has multiple coexisting periodic or chaotic attractors or a mixture of both. It is hence appropriate to understand how a probability ensemble might behave in such a system. The details of the dependence on the ensemble are particularly relevant to the issue of the possible experimental work on these issues, since experiments are always conducted, by virtue of finite precision, over finite time and finite ensembles. It is therefore important to probe the Barbi and Salerno results with regard to the ensemble used, and more formally, to see how nonergodicity alters our considerations about the current, as we do in this paper.

We report here on studies on the properties of the current in a chaotic deterministic ratchet, specifically the same system as considered by Mateos [8] and Barbi and Salerno [9]. We show that single-trajectory analysis is inherently flawed in this context especially because of the existence of multiple attractors. We then consider the impact of different kinds of ensembles of particles on the current and show that the current depends significantly on the details of the initial ensemble. We also show that it is important to discard transients in quantifying the current, therefore emphasizing that since broad heuristics are rare in chaotic systems, it is critical to understand the ensemble dependence of the transport properties of chaotic ratchets. We then proceed to discuss the connection between the bifurcation diagram for individual particles and the behavior of the current. We find that while we disagree with Barbi and Salerno's analysis and hence many of their results, the broader conclusion still holds. That is, it is indeed possible to have current reversals in the absence of bifurcations from chaos to periodic behavior as well as bifurcations without any accompanying current reversals even when the analysis is extended correctly to ensembles. The result of our investigation is therefore that the transport properties of a chaotic ratchet are not as simple as the initial conjecture. However, we do find evidence for a generalized version of Mateos's conjecture. That is, in general, bifurcations for trajectory dynamics as a function of system parameter seem to be associated with abrupt changes in the current. Depending on the specific value of the current, these abrupt changes may lead the net current to reverse direction, but not necessarily so.

We start below with a preparatory discussion necessary to understand the details of the connection between bifurcations and current reversal, where we discuss the potential and phase space for single trajectories for this system, where we also define a bifurcation diagram for this system. In the next section, we discuss the subtleties of establishing a connection between the behavior of individual trajectories and of ensembles. After this, we are able to compare details of specific trajectory bifurcation curves with current curves, and thus justify our broader statements above, after which we conclude.

## II. REGULARITY AND CHAOS IN SINGLE-PARTICLE RATCHET DYNAMICS

To discover heuristic rules we argue that we must consider specific systems in great detail before generalizing. We choose the same one-dimensional ratchet previously studied [8,9]. We consider an ensemble of particles moving in an asymmetric periodic potential, driven by a periodic time-dependent external force, where the force has a zero time average. There is no noise in the system, so it is completely deterministic, although there is damping. The equations of motion for an individual trajectory for such a system are given in dimensionless variables by

$$\epsilon \ddot{x} + b\dot{x} + \frac{dV(x)}{dx} = a\cos(\omega t), \tag{1}$$

where the periodic asymmetric potential can be written in the form

$$V(x) = C - \frac{1}{4\pi^2\delta}\left(\sin[2\pi(x-x_0)] + \frac{1}{4}\sin[4\pi(x-x_0)]\right). \tag{2}$$

In this equation $C$, $x_0$ have been introduced for convenience such that one potential minimum exists at the origin with $V(0)=0$ and the term $\delta=\sin(2\pi|x_0|)+\frac{1}{4}\sin(4\pi|x_0|)$. The constant $\epsilon$ is introduced for convenience in discussing the difference between the overdamped and underdamped cases in the next section. We note that the dimensionality of the variables means that all quantities computed and reported in our work are dimensionless.

The undamped undriven ratchet—corresponding to the unperturbed potential $V(x)$—looks like a series of asymmetric pendula. That is, individual trajectories have one of following possible behaviors as depicted in Fig. 1(a): (i) Inside the potential wells, trajectories and all their properties oscillate. Outside the wells, the trajectories either (ii) librate to the right or (iii) to the left, depending upon initial conditions. There are also (iv) trajectories on the separatrices between the oscillating and librating orbits, moving between unstable fixed points in infinite time, as well as the unstable and stable fixed points themselves, all of which constitute a set of negligible measure. A nonzero $b$-dependent damping term in Eq. (1) makes the stable fixed points the only attractors for the system. When the driving is turned on, the phase space becomes chaotic with the usual phenomena of intertwining separatrices and resulting homoclinic tangles. Individual trajectories are now very complicated, depending sensitively on the choice of parameters and initial conditions. We show snapshots of the development of this chaos in the set of Poincaré sections in Figs. 1(b) and 1(c) together with a period-four orbit represented by the center of the circles.

A broad characterization of the dynamics of the problem as a function of a parameter ($a$, $b$ or $\omega$) emerges in a bifurcation diagram. This can be constructed in several different and essentially equivalent ways. The relatively standard form that we use proceeds as follows: First choose the bifurcation parameter (let us say $a$) and correspondingly choose fixed values of $b,\omega$, and start with a given value for $a=a_{\min}$. Now iterate an initial condition, recording the value of the particle's position $x(T_P)$ at times $T_p$ from its integrated trajectory [sometimes we record $v(T_P)=\dot{x}(T_P)$]. This is done stroboscopically at discrete times $T_P=n_pT_\omega$ where $T_\omega=2\pi/\omega$ and $n_P$ is an integer $1\leq n_P<M$ with $M$ the maximum number of observations made. Of these, discard observations at times less than some cutoff time $n_cT_\omega$ and plot the remaining points against $a_{\min}$. It must be noted that discarding transient behavior is critical to get results which are independent of initial condition, and we shall emphasize this further below in the context of the net transport or current. If the system has a fixed-point attractor then all of the data lie at one particular location $x_c$. A periodic orbit with period $jT_\omega$ (that is, with

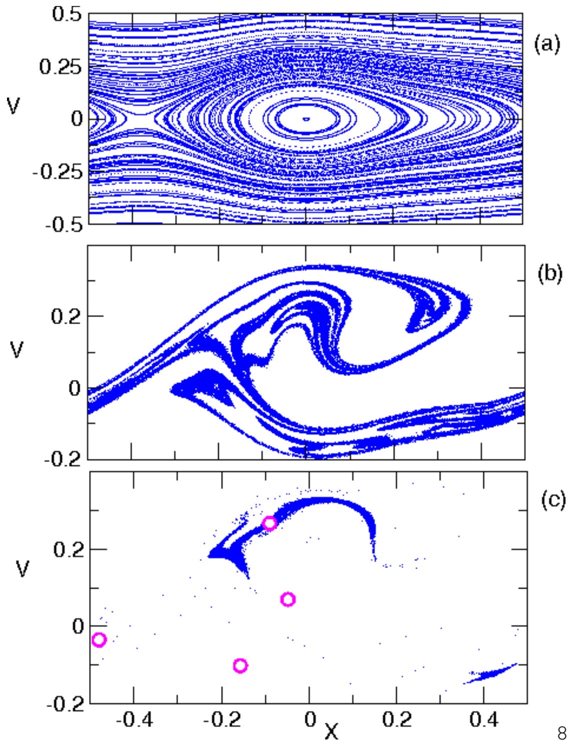


FIG. 1. (Color online) (a) Classical phase for the unperturbed system. For $\omega=0.67$, $b=0.1$, and $\epsilon=1$, two chaotic attractors emerge with (b) $a=0.11$ and (c) $a=0.155$. Circles superposed in (c) represent a period four attractor obtained with $a=0.08125$.

period commensurate with the driving) shows up with $M-n_t$ points occupying only $j$ different locations of $x$ for $a_{\rm min}$. All other orbits, including periodic orbits of incommensurate period result in a simply-connected or multiply-connected dense set of points. For the next value $a=a_{\rm min}+da$, the last computed value of $x,v$ at $a=a_{\rm min}$ are used as initial conditions, and previously, results are stored after cutoff and so on until $a=a_{\rm min}+(j-1)da=a_{\rm max}$. That is, the bifurcation diagram is generated by sweeping the relevant parameter, in this case $a$, from $a_{\rm min}$ through some maximum value $a_{\rm max}$. With this procedure, it turns out to be critical to sweep the system both ways, from $a_{\rm max}$ to $a_{\rm min}$ since there is some hysteresis in the system as well, a point we shall discuss further below. This procedure is intended to catch all coexisting attractors of the system with the specified parameter range. We note that bifurcation diagrams can be constructed using either $x$ or $v$: Chaos will be visible at the same place in either variable; and this is usually true for periodic orbits as well, with the exception of unbounded trajectories of a certain kind [15].

Having broadly understood the wide range of behavior for individual trajectories in this system, we now turn in the next section to a discussion of the nonequilibrium properties of a statistical ensemble of these trajectories, specifically the current for an ensemble.

## III. ENSEMBLES AND CURRENTS FOR A ROCKED RATCHET SYSTEM

Now let us consider the notion of a "current" for a driven ratchet described by Eq. (1). As a preliminary remark, note that regular trajectories for this are given by

$$x(t+nT)=x(t)+2\pi m, \tag{3}$$

where $(n,m)\in(\mathbb{N},\mathbb{Z})$, and $T=2\pi/\omega$ represents the driver period. It turns out that for $(n,m=0)$, the trajectories of the system are nontransporting while the case $(n,m\neq 0)$ leads to transporting ones. The asymptotic velocity in this case is constant, independent of initial conditions and can be expressed as [7]

$$v_{nm}=\frac{x(t+nT)-x(t)}{nT}=\frac{2\pi m}{nT}=\frac{m}{n}\omega. \tag{4}$$

Depending on the system parameters, the motion can be bound $(v_{nm}=0)$ or unrestricted with a nonzero velocity $(v_{nm}\neq 0)$ which then represents the net current flowing through the system. For the special case of an overdamped system $(\epsilon=0)$, the dynamics is purely regular. For an example of such a system with more than one attractor, which are, however, zero-current attractors, see Ref. [15].

The underdamped system ($\epsilon\neq 0$, and without loss of generality $\epsilon=1$) exhibits both regular and chaotic behavior. Further, the system may support more than one periodic and/or chaotic attractor each with distinct basis of attractions. This possible coexistence of attractors makes the computation of the current nontrivial, and as we expand below, requires use of ensembles for the computation. We list the different classes of trajectories, with distinct transport properties.

(1) *Regular trajectories*. Regular trajectories are of course characterized by the velocity [Eq. (4)] or simply by the tuples $(n,m)$. If the system contains only one periodic attractor, we know that (i) nontransporting trajectories will be characterized by $(n,0)$ leading to zero current, while (ii) transporting trajectories are characterized by $(n,m\neq 0)$ leading to a net current. However when there are $K$ attractors in the system corresponding to different sets $(n_k,m_k)$ with $k=\{1,2,\ldots,K\}$, the net drift must be averaged over these different velocities $(v_{n_k,m_k})$.

(2) *Chaotic trajectories*. If the system supports only one chaotic attractor, then two situations are possible. (i) No net drift: The average velocity on the attractor, with the particle moving back and forth, is zero, or (ii) Net drift: This can be either to the left or to the right, i.e., particles move on average to the left or to the right, whence the resulting drift is expected to exhibit strong fluctuations [7]. Given multiple attractors with complicated basins of attraction, the notion of transport is clearly well-defined only in the average, particular as systems typically have coexisting chaotic and regular attractors. This situation requires careful analysis with ensemble-based statistical computations, emphasized by the nonergodicity of the system.

As a result, the analysis of Barbi and Salerno [9] is immediately seen as flawed. In particular, they assume *no transport* is possible for a chaotic behavior (see the middle of the first column, p. 1989 of Ref. [9]). This intuitive physical picture of theirs is only valid if the system exhibits regular behavior, and more specifically with only one periodic attractor (see discussion of regular behavior above). In particular, in this situation of regular behavior with a net drift, the motion becomes effectively locked to the driver. When more than one attractor is present, this is no longer true. Further, if one looks carefully at their plots, one can identify some chaotic areas with nonzero velocity, meaning that there is transport for chaotic behavior; this in fact makes their paper self-contradictory. In fact, the system under consideration supports many attractors for some parameters. This is why when Barbi and Salerno change the initial condition, they see a change from zero current to a net drift (see their Fig. 4). This can also be observed in their Fig. 5 where hysteresis is highlighted. They ascribe the origin of current reversals to the different stability properties of the periodic rotating orbits. However, it is clear that this can be understood as a consequence of using only a single trajectory despite the presence of multiple attractors in the system.

Thus, the net flux of the system is correctly computed with a suitably broad distribution [7,8]. The current $J$ is then defined as follows: An ensemble average is performed at a given observation time $t_j$ to yield the average velocity as (note that for $\epsilon=1$ the momentum is the same as the velocity, $v\equiv p$)

$$v_j = \int dp\, dx\, p(t_j)\rho(x,p). \tag{5}$$

For any finite ensemble (relevant both computationally and experimentally) this goes over to the intuitive definition of Mateos [8]:

$$v_j = \frac{1}{N}\sum_{i=1}^{N} \dot{x}_i(t_j). \tag{6}$$

This average velocity is then further time-averaged; for discrete time $t_j$ for observation this leads to

$$J = \frac{1}{M}\sum_{j=1}^{M} v_j, \tag{7}$$

where $M$ is the number of time observations made. A further parameter dependence that is being suppressed here is the shape and location of the ensemble being used.

In the case of multiple periodic attractors we may write the expression for the current as

$$J = \int dp\, dx \sum_{i}^{N} v_i I_i[\rho(x,p)], \tag{8}$$

where $I_i$ is the indicator function equalling unity if the point $(x,p)$ belongs to a basin of attraction of $i$th attractor and is zero otherwise, and $v_i$ is the attractor's winding number. Clearly different initial distributions will yield different asymptotic current values $J$ even for this situation. For a chaotic system, the transport properties of an ensemble are even more strongly dependent on the part of the phase-space being sampled. It is therefore important to consider many different initial conditions to generate a current.

The first straightforward result we show in Fig. 2 is that in the case of chaotic trajectories, a single trajectory easily displays behavior very different from that of many trajectories, although in the regular regime a single trajectory yields essentially the same result as obtained from many trajectories.

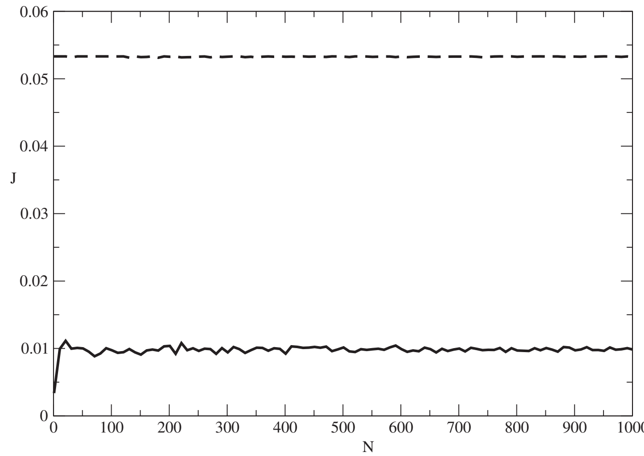


FIG. 2. Current $J$ versus the number of trajectories $N$ for $\omega=0.67$, $b=0.1$; dashed lines correspond to regular motion with $a=0.12$ while solid lines correspond to chaotic motion with $a=0.08$. Note that a single trajectory is sufficient for a regular motion while the convergence in the chaotic case is only obtained if the $N$ exceeds the threshold, $N\geq N_{\rm thr}=100$.

Further consider the current in Fig. 3 where we superimpose the different curves resulting from varying the number of points in the initial ensemble. First, the curve is significantly smoother as a function of $a$ for larger $N$. Even more relevant is the fact that the single trajectory data ($N=1$) may show current reversals that do not exist in the large $N$ data. Also, note that single-trajectory current values are typically significantly greater than ensemble averages since an arbitrarily chosen ensemble has particles with behaviors which often average out. However, it is not true that only a few trajectories dominate the dynamics completely, else there would not be a saturation of the current as a function of $N$. All this is clear in Fig. 3.

The current also depends on the location of the centroid $\langle x\rangle,\langle p\rangle$ of the initial ensemble, particularly for small $N$ and trivially for $N=1$ given a nonergodic and chaotic system.

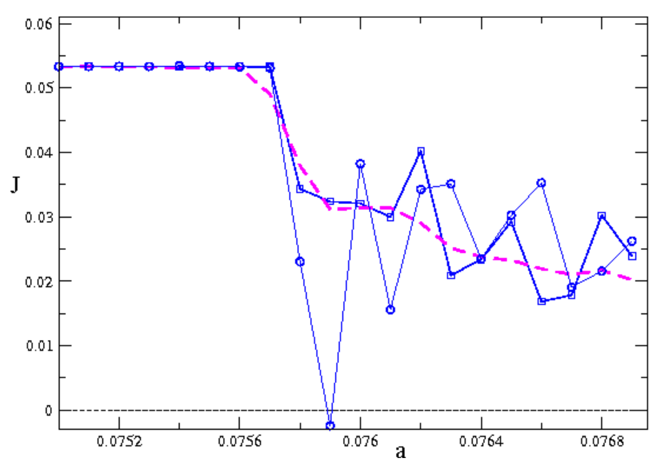


FIG. 3. (Color online) Current $J$ versus $a$ for different set of trajectories $N$; $N=1$ (circles), $N=10$ (square), and $N=100$ (dashed lines). Note that a single trajectory suffices in the regular regime where all the curves match. In the chaotic regime, as $N$ increases, the curves converge toward the dashed one.

Further, considering a Gaussian ensemble, say, the width $W$ of the ensemble also affects the details of the current, and can hide or show false information about current reversal, as seen in at $a\approx 0.15$ in Fig. 6(a) for example. Notice further that in Fig. 5, at $a\approx 0.065$ and Fig. 6, at $a\approx 0.15$, the deviations between the different ensembles is particularly pronounced. These points are close to bifurcation points where some sort of symmetry breaking is clearly occurring, which underlines our emphasis on the relevance of specifying ensemble characteristics in the neighborhood of unstable behavior. Moreover, these specific bifurcations are associated with multiple periodic attractors with disjoint basins of attraction as is clear from the fact that scanning up in $a$ produces a different bifurcation diagram than scanning down in $a$ does. This makes it clear why these regions stand out among all the bifurcations in the parameter range shown.

To recap: For Hamiltonian ($b=0$) motion, we know that there is a typical structure of stable islands embedded in a chaotic sea which have quite complicated behavior [13], and the current always depends on the location of the initial conditions. However, when the damping is turned on the phase space consists in general of attractors, so if transient behavior is appropriately discarded, the dependence on location or spread of the initial conditions can be removed. In particular, in the chaotic regime of a non-Hamiltonian system, the initial ensemble needs to be chosen to span the entire phase space, to ensure convergence. This is also valid for a regular system of more than one attractor. However, in the regular regime with only one attractor, it is not important to take a large ensemble and a single trajectory can suffice (again, transients must be discarded). The transient time depends on system parameters but is usually several times the driving period $T$. To account for it, the definition of the current needs to be modified to

$$J=\frac{1}{M-n_c}\sum_{j=n_c}^{M}v_j, \qquad (9)$$

where $n_c$ is some empirically obtained cutoff such that we get a converged current (for instance, in our calculations, we obtained converged results with $n_c=10^5$, $M=10^7$).

Finally, we reemphasize here the point that currents computed via Eq. (9) do *not* converge to those computed by Barbi and Salerno even if all transients are discarded. This is clear in Fig. 4 where we show results for a *single* trajectory used in Eq. (9). Although this disagrees as expected with the multiple trajectory calculations in Fig. 5, more importantly it also disagrees with the "current" computed in the manner of Barbi and Salerno. In particular, given that there are at least two attractors in the system, the current computed by letting a single trajectory find the attractor, as in Eq. (9), gives us the jagged curve in the range $a\in\{0.0625,0.07\}$, emphasizing (a) why the Barbi and Salerno phase-locking based analysis is erroneous independent of issues of transience, as well as (b) why ensembles should be used in general.

Armed with this background, we are now finally in a position to compare bifurcation diagrams with the current, as we do in the next section.

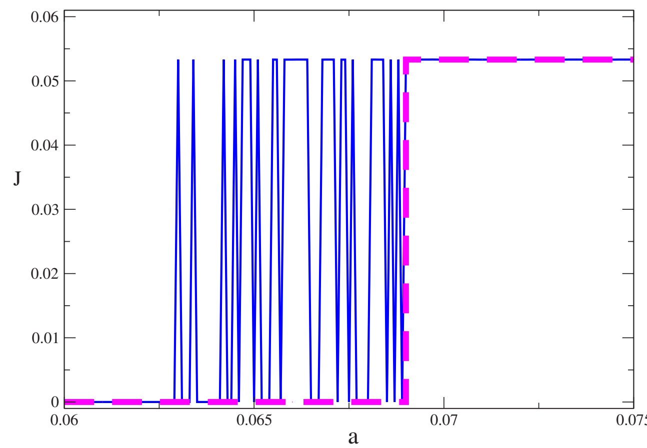


FIG. 4. (Color online) Current $J$ versus $a$ for a single trajectory ($N=1$), with $n_c=5\times 10^5$ time steps discarded from a total $M=10^7$ time steps (blue) showing many transitions between the upper and lower values, compared to Barbi and Salerno's results (dashed in magenta) showing a single transition. Here $\omega=0.67$ and $b=0.1$.

## IV. RELATIONSHIP BETWEEN BIFURCATION DIAGRAMS AND ENSEMBLE CURRENTS

Our results are presented in Figs. 5 and 6, in which we plot both the ensemble current and the bifurcation diagram as a function of the parameter $a$. Our observations are labeled with the letters (a)–(h), but the main points are distilled into heuristic statements we label below with Roman numerals (I)–(IV).

We now compare our results against the specific details of Barbi and Salerno's treatment. In particular, we look at their Figs. 2, 3(a), and 3(b), where they scan the parameter region $b=0.1$, $\omega=0$, 67, and $a\in(0.6,0.24)$. We note first that (a) our bifurcation diagrams as shown in Figs. 5(c) and 6(c) differ from theirs. This is easily understood as a consequence of "hysteresis." Recall our discussion of computing the bifurcation diagram by using the final condition from the previous parameter value while scanning the parameter $a$, and also recall that we emphasized that it was important to scan in both directions in $a$. This is because the disjoint basins of attractions when a system is bifurcating to multiple attractors make it possible to entirely miss some attractors in the system if this is not done, and this is what seems to have happened with Barbi and Salerno.

Now turning to the current itself, our computations with larger numbers of particles $N$ yields different results, as we show in the recomputed versions of their figures, presented here in Figs. 5 and 6. Specifically, (b) the single-trajectory results are cleaner, while the ensemble results, even when converged, show statistical fluctuations. We clarify here how we distinguish between current "fluctuations" and "abrupt" current changes. In the absence of any motivation to be more complicated, we have looked at changes in the current $\Delta J$ which happen over a smaller range of the bifurcation parameter $a$ than the local trend in $J$ (that is, where the absolute slope is significantly greater than the average slope). We have then used the naïve definition that a change in the current is considered a fluctuation when it is substantially

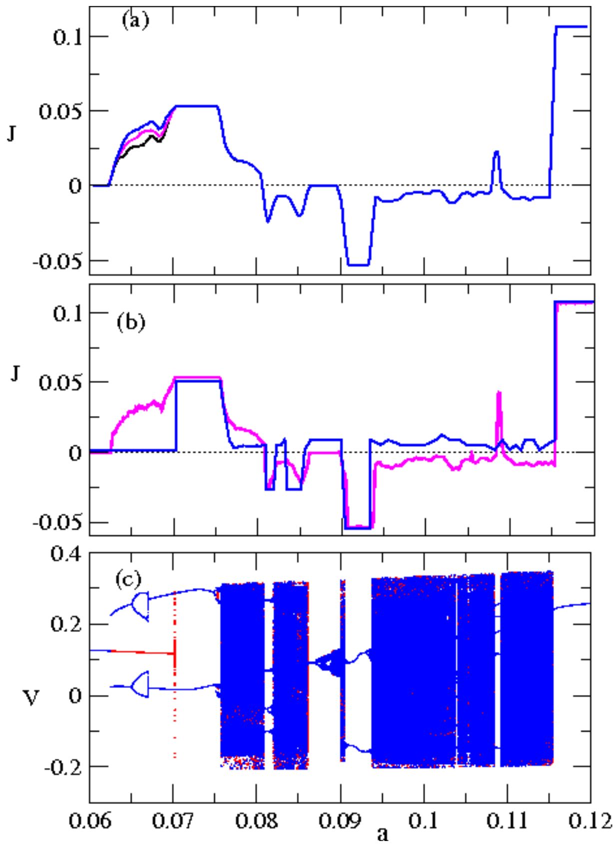


FIG. 5. (Color online) Currents $J$ and corresponding bifurcation diagrams $v$ versus $a$. In (a) $J$ is computed with an ensemble of trajectories of different root-mean-square Gaussian width $W$, starting centered either at the stable fixed point $(0,0)$ or at the unstable fixed point $(-0.375,0)$. The blue (topmost) and the black (lowermost) curves are obtained with trajectories centered at $(0,0)$, for $W=0.25$ and with $W=0.5$, respectively, while the magenta (middle) is obtained with trajectories centered at $(-0.375,0)$ and $W=0.25$. In (b) we compare our converged results from (a) (shown in magenta) obtained with trajectories centered at $(0,0)$ and covering the entire space, $W=1$, to the single trajectory results (shown in blue) of Barbi and Salerno. In (c) the bifurcation diagram (blue) is obtained with increasing $a$ while the bifurcation diagram (red) is obtained with decreasing $a$. Here $\omega=0.67$, $b=0.1$, $N=10^3$, $n_c=5\times10^5$, and $M=10^7$.

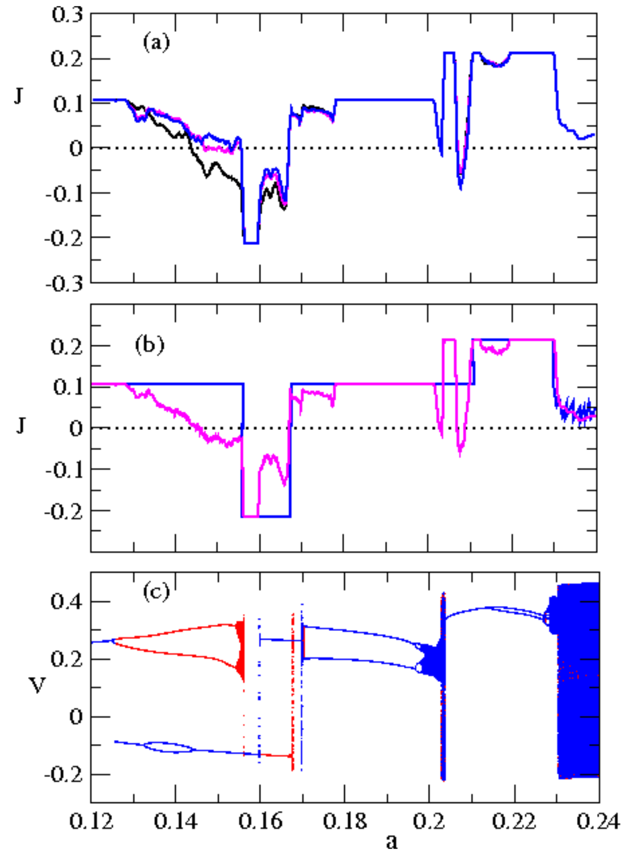


FIG. 6. (Color online) Same as Fig. 5 except for the range of $a$ considered.

smaller than the "local average." When $\Delta J$ is of the order of $J$ or greater, we consider this a relevant change, and classify it as "abrupt" or "sudden."

With all this the ensemble results are broadly consistent with Barbi and Salerno's disagreement with the Mateos conjecture, although different in several important details. For instance, (c) the bifurcation at $a\approx0.07$ has a much gentler impact on the ensemble current, which has been growing for a while, while the Barbi-Salerno result changes abruptly. This growing ensemble current is correlated with the appearance of several other periodic attractors that were missed by Barbi-Salerno. In the range $a\in(0.075,0.095)$ (d) the single-trajectory current actually seems to validate the Mateos conjecture with several current reversals coinciding with period-doubling bifurcations leading to order-chaos transitions, such in the approximate ranges $a\in(0.075,0.076)$, $(0.08,0.082)$, and $(0.086,0.09)$. However, there is only one instance of current-reversal for the ensemble current, at $a\approx0.08$, even though the current shows abrupt changes as a function of parameter. It is easy to find many examples like this, leading to the negative conclusion that (I) *not all bifurcations lead to current reversal*. Consider also (e) the interesting fact that the single-trajectory current completely misses the bifurcation-associated spike at $(a\approx0.11)$.

At this point we have not precluded the more restricted statement that all current reversals are associated with bifurcations. However, in Fig. 6 we see (f) current reversals, at $a\approx0.15$, 0.205, and 0.21 that do not seem to be associated with any bifurcations whatsoever. Note that the difference between the single-trajectory current and the ensemble-computed current in the range $a\leq0.17$ can be understood by recognizing the presence of alternate attractors that the Barbi-Salerno scan had missed. On the other hand, (g) there is a jump in the ensemble current at $a\approx0.18$ that does not seem related to any bifurcation at all. Finally, (h) apart from the current reversals mentioned in (g), the ensemble current shows other structure in the range $a\in\{0.2,0.22\}$ that does not seem to be associated with bifurcations. The single-current calculation almost entirely ignores this, except for one jump at $a\approx0.21$.

All of this is summarized in two statements: The first is a reiteration of the fact that there is significant information in the ensemble current that cannot be obtained from the single-trajectory current. The second is again a negative heuristic, that (II) *not all current reversals are associated with bifurcations*. One possible way of retaining the Mateos conjecture is to weaken it, into the statement that (III) *most current reversals are associated with bifurcations*. However, a *different* rule of thumb, previously not proposed, emerges from our studies. This generalizes Mateos' conjecture to say that (IV) *bifurcations correspond to sudden current changes (spikes or jumps)*. Note that this means these changes in current are not necessarily reversals of direction. If this current jump or spike goes through zero, this coincides with a current reversal, making the Mateos conjecture a special case. The physical basis of this argument is the fact that ensembles of particles in chaotic systems *can* have net directed transport but the details of this behavior depends relatively sensitively on the system parameters. This parameter dependence is greatly exaggerated at the bifurcation point, when the dynamics of the underlying single-particle system undergoes a transition—a period-doubling transition, for example, or one from chaos to regular behavior. Scanning the relevant figures, we see that this is a very useful rule of thumb, significantly enhancing our ability to characterize changes in the behavior of the current as a function of parameter.

In conclusion therefore, the results of this paper could be summarized as the negative result that even when the flawed analysis of Barbi and Salerno is corrected to the correct ensemble methods, the Mateos conjecture does not always hold. However, we have taken the approach that it is useful to find general rules of thumb (even if not universally valid) to understand the complicated behavior of nonequilibrium nonlinear statistical mechanical systems. In the case of chaotic deterministic ratchets, we have shown that it is important to factor out issues of size, location, spread, and transience in computing the "current" due to an ensemble before we search for such rules, and that the dependence on ensemble characteristics is most critical near certain bifurcation points. We have then argued that the following heuristic characteristics hold: Changes in the dynamics seen in bifurcation diagrams for a system correspond to sudden spikes or jumps in the current for an ensemble in the same system. Current reversals are a special case of this. However, not all spikes or jumps correspond to a bifurcation, nor vice versa. The open question is clearly to figure out if the reason for when these rules are violated or are valid can be made more concrete.

## ACKNOWLEDGMENTS

A.K. gratefully acknowledges T. Barsch and Kamal P. Singh for stimulating discussions, and financial support from the Alexander von Humboldt foundation as well as the Max-Planck Gesellschaft through the Reimar Lüst fund (2005). A.K.P. is grateful to Carleton College for sabbatical leave support through the "SIT, Wallin, and Class of 1949" Funds, and to the MPIPKS for hosting him for a sabbatical visit, which led to this collaboration. Discussions with J.-M. Rost on preliminary results and very useful detailed comments from Peter Hänggi are also gratefully acknowledged.

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